is stabilizable and we can choose P_* so as to fulfill the conditions

$$P_* \supset P^\alpha, \quad Q_* \supset Q^\alpha, \quad Q = \varkappa P_* \quad (0 < \varkappa < 1)$$

We set $x = y - z$ and we write the model's equation as

$$dw \,/\, dt = Aw + Bu_* - Bv_*, \quad u_* \in P_*, \quad v_* \in Q$$

If the initial position $\{t_0, y_0, z_0\}$ is such that it is impossible to bring system (2.14) into the ε-neighborhood of point $s = 0$ in finite time by a choice of control $m \in (1 - \varkappa)\, P_*$, then to retain the position $\{t, w[t]\}$ on bridge $W_{\varepsilon_0}^\infty$ it is sufficient to choose v_* such that $u_* - v_* \in (1 - \varkappa)\, P_*$. Thus, in the given example all the needed constructions connected with the bridge $W_{\varepsilon_\iota}^\infty$ turn out to be very simple, although the description of the bridge itself remains unknown.

The author thanks N. N. Krasovskii for posing the problem and for valuable advice.

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QUALITATIVE INVESTIGATION OF A PIECEWISE LINEAR SYSTEM

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We use the methods of the theory of bifurcation and piecewise linear approximation to the characteristic with a falling segment, in the qualitative investigation of a system which is of practical interest. We trace the possible bifurcations and follow the behavior of the bifurcation curves. The system has been studied by a number of authors, using various approximations [1 – 9], however none of them gave a complete qualitative investigation.

1. Equations of motion. We consider the system

$$x^{\cdot} = y - \varphi(x), \quad y^{\cdot} = L, \quad L \equiv \sigma - \lambda x - y \quad (\sigma > 0,\ \lambda > 0) \qquad (1.1)$$

where $\varphi(x)$ is a nonlinear function containing a falling segment. Let us approximate $\varphi(x)$ with a piecewise linear function consisting of three linear segments: a falling segment of slope $k = -\alpha_2 < 0$, and two rising segments of slope $k = \alpha_1 > 0$. Under this approximation the phase space splits into three regions in each of which the system is linear. Regions I and III contain the rising branches of the characteristic and region II contains the falling segment (Fig. 1).

2. States of equilibrium. Partitioning of the parameter space according to the number and character of the states of equilibrium. Either one, or three coarse states of equilibrium are possible. In the case of a single state of equilibrium we have a focus (node) which is always stable in the regions I and III and unstable in II, provided that $\alpha_2 > 1$. In the case of three states of equilibrium we have foci (nodes) in the regions I and III and a saddle in the region II, and they are always stable. The straight line segments $\sigma = x_1\lambda + y_1$ and $\sigma = x_2\lambda + y_2$ (x_1, y_1 and x_2, y_2 are the coordinates of the corner points of the characteristic) form in the plane $\lambda\sigma$ for $\lambda \leqslant \alpha_2$ a discriminant curve separating the region of three states from the region of a single state of equilibrium. The points lying on the discriminant curve have the corresponding matched state of equilibrium of the saddle-focus or saddle-node type, while the corner point ($\lambda = \alpha_2$) has the corresponding unstable stationary segment coinciding with the falling segment of the characteristic. When $\alpha_2 < 1$, we have no closed trajectories and the only possible bifurcations correspond to the appearance and disappearance of the states of equilibrium. In what follows, we shall concern ourselves with the case $\alpha_2 > 1$ and $(\alpha_1 - 1)^2 < 4\alpha_2$ which admits various types of bifurcations.

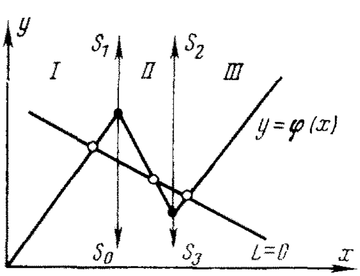

Fig. 1

3. Bifurcations of the states of equilibrium. 3.1. Stability of the state of equilibrium on the matching line. Let the line $\sigma - \lambda x - y = 0$ pass through the corner point (x_1, y_1) of the characteristic at the boundary separating the regions I and II, and let $\lambda > {}^1/_4\,(\alpha_2 + 1)^2 > \alpha_2$. Then the region I will be filled with segments of the trajectories of the stable focus, and the region II with the trajectories of the unstable focus. Let us introduce the positive coordinates s_0 and s_1 on the line joining the regions I and II (Fig. 1). The transformations $s_0 \to s_1$ along the trajectories of the region I and $s_1 \to \bar{s}_0$ along the trajectories of the region II can be written as

$$s_1 = s_0 \exp[-h_1\pi / \omega_1], \qquad \bar{s}_0 = s_1 \exp[-h_2\pi / \omega_2] \qquad (3.1)$$

Here ω_i and $-h_i$ $(i = 1,2)$ denote the imaginary and the real parts of the roots of the characteristic equation for the regions I and II, respectively. The state of equilibrium will be represented by a matched center ($\bar{s}_0 \equiv s_0$), if $h_1 / \omega_1 + h_2 / \omega_2 = 0$ or, in the expanded form, if

$$\lambda = \lambda^{+} \equiv (\alpha_1\alpha_2 + 1)\,(\alpha_1 - \alpha_2 + 2)^{-1}$$

The focus on the matching line will be stable ($\bar{s}_0 < s_0$) when $\lambda > \lambda^+$, and unstable ($\bar{s}_0 > s_0$) when $\lambda < \lambda^+$.

3.2. Emergence of a limit cycle from the focus-type state of equilibrium when the latter passes through the matching line. We shall show that not more than a single limit cycle can exist in the regions I and II. Let us consider the transformation $s_0 \to \bar{s}_0$ along the trajectories of the regions I and II. For region I we have

$$s_0 = \frac{\delta_0}{\sin \omega_1\tau_1} [\omega_1 \cos \omega_1\tau_1 + h_1 \sin \omega_1\tau_1 - \omega_1 e^{h_1\tau_1}] = \delta_0 \zeta(\tau_1) \tag{3.2}$$

$$s_1 = \frac{\delta_0}{\sin \omega_1\tau_1} [\omega_1 \cos \omega_1\tau_1 - h_1 \sin \omega_1\tau_1 - \omega_1 e^{-h_1\tau_1}] = \delta_0 \chi(\tau_1)$$

where δ_0 is the distance from the boundary separating regions I and II, to the state of equilibrium, while χ and ζ are monotonic functions (which increase or decrease, depending on the sign of δ_0). The transformation along the trajectories of the region II can be written out in an analogous manner.

Computing the derivative of the successor function, we obtain

$$d\bar{s}_0 / ds_0 = s_0 \bar{s}_0^{-1} \exp [-2(h_1\tau_1 + h_2\theta)] \tag{3.3}$$

$$h_1 = (1 + \alpha_1)/2 > 0, \qquad h_2 = (1 - \alpha_2)/2 < 0$$

where τ_1 and θ denote the times of motion along the trajectories of the regions I and II, respectively.

Let the state of equilibrium lie in the region I. Then for the periodic solution $\bar{s}_0 = s_0$ the time τ_1 decreases to the value π / ω_1), with increasing s_0, the time θ increases to π / ω_2), and the derivative (3.3) grows. For this reason the successor function can intersect the bisectrix at not more than two points and the stationary point with the smaller coordinate must be stable, while that with the larger coordinate must be unstable. Since it was assumed that the state of equilibrium lies in the region I and is a stable focus which cannot however be encircled by a stable cycle, therefore in the regions I and II we cannot have more than one cycle and this cycle must be unstable.

Let now the state of equilibrium lie in the region II. Then the time τ_1 increases with increasing s_0, while θ decreases. In an analogous manner we find, that in this case we cannot have more than one stable limit cycle.

Let the line $L = 0$ pass through the upper corner point of the characteristic. We shall consider two cases.

1). $\lambda > \lambda^+$. The matched focus is stable. The trajectory passing through the lower corner point winds, by virtue of (3.1), towards the state of equilibrium as $t \to \infty$ This trajectory remains spiral under small displacements of the line $L = 0$. If after such a displacement the state of equilibrium enters the region II, it becomes unstable and as a consequence, at least one stable limit cycle appears. As we said before, this cycle is unique. Let the state of equilibrium enter the region I as the result of a displacement. Since in the combined region I and II not more than one cycle can exist and the focus remains stable, therefore no cycles appear.

2). $\lambda < \lambda^+$. In the analogous manner we find that if, as the result of a small displacement, the state of equilibrium enters the region II, then no cycles appear, while the entry into the region I is accompanied by the appearance of an unstable cycle.

3.3. Emergence of limit cycles (single or double) from the boundary of the region filled with closed trajectories. Consider the transformation $\bar{s}_0 = f(s_0)$ consisting of two segments: $\bar{s}_0 = \varphi(s_0)$ along the trajectory of the regions I and II and $\bar{s}_0 = \psi(s_0)$ in all regions. We shall show that $f(s_0)$ is differentiable at the matching point of the two segments. The transformation $s_0 \to s_1$ along the trajectories of the region I is given by (3.2), and the transformations $s_1 \to s_2$, $s_2 \to s_3$ and $s_3 \to s_0$ can be written in the same manner. The value of $d\bar{s}_0 / ds_0$ for the function $\varphi(s_0)$ is given in (3.3), and for the function $\psi(s_0)$ it is

$$d\bar{s}_0 / ds_0 = s_0\bar{s}^{-1} \exp\left[-2h_1(\tau_1 + \tau_3) - 2h_2(\tau_2 + \tau_4)\right] \tag{3.4}$$

Here τ_1 and τ_3 denote the times of motion in the regions I and III, while τ_2 and τ_4 denote the times of motion in the upper and lower part of the region II. Let $s_0 = s_0^*$ be a limiting value separating the intervals of definition of the transformations $\varphi(s_0)$ and $\psi(s_0)$. The derivatives for φ and ψ coincide at the matching point; when $s_0 = s_0^*$ we have $\tau_3 = 0$, $\theta = \theta^*$ and $\tau_2 + \tau_4 = 0^*$. Let now the line $L = 0$ pass through the corner point x_1, y_1 of the characteristic and $\lambda = \lambda^+$. We shall show that there are no limit cycles.

The successor function at the $s_0\bar{s}_0$-plane consists of a segment of the bisectrix $\bar{s}_0 = s_0 < s_0^*$ and the curve $\bar{s}_0 = \psi(s_0)$. The function $s_0 = f(s_0)$ is differentiable at the matching point, consequently at $\lambda = \lambda^+$ we have $d\bar{s}_0 / ds_0 = 1$ (from (3.4) we also obtain that $d^2\bar{s}_0 / ds_0^2 < 0$). When the value of s_0 increases from s_0^*, the exponential index in (3.4) decreases monotonously from its zero value at the matching point ($\tau_1 =$ const, τ_3 increases and $h_1 > 0$; τ_2 and τ_4 decrease and $h_2 < 0$). The curve $\bar{s}_0 = \psi(s_0)$ has a single point $s_0 > s_0^*$ of intersection with the bisectrix and no other points of intersection (or contact) are possible. The curve for $s_0 = s_0^*$ is situated below the bisectrix. The spirals constructed by joining the trajectories in the regions I – III, wind onto the boundary of the region filled with the closed curves matched from the trajectories in the regions I and II.

Under a small variation of the parameters σ and λ the successor function of the altered system lies in a close vicinity of the successor function of the initial system. On moving along the half-line $L_1 = 0$ ($L_1 \equiv \sigma - \lambda x_1 - y_1 = 0$, $\lambda > \alpha_2$) from the value $\lambda = \lambda^+$ in the direction of decreasing λ we find, that the successor function for $s_0 < s_0^*$ will be represented by a straight line passing through the coordinate origin above the bisectrix, and for $s_0 \geqslant s_0^*$ by the curve $\bar{s}_0 = \psi(s_0)$ intersecting the bisectrix once (at the matching point $d^2\bar{s}_0 / ds_0^2 \neq 0$ when $\lambda = \lambda^+$ and $\sigma = \sigma^+$). A unique, stable limit cycle appears at the boundary of the region filled with the closed curves. If we now decrease the value of σ, the initial point of the successor function displaces from the coordinate origin along the axis s_0 (the smallest value of s_0 corresponds to the trajectory entering the stable focus and tangent to the matching line at $\bar{s}_0 = 0$), and the successor function $\bar{s}_0 = f(s_0)$ will intersect the bisectrix twice (a unique unstable limit cycle emerges from the focus as the latter is displaced from the matching line). If we now move along the half-line from $\lambda = \lambda^+$ in the direction of increasing λ and reduce the value of σ, then the successor function will wholly lie below the bisectrix. From the properties of continuity and differentiability of the successor function it follows that in any small semi-neighborhood of the point λ^+, σ^+ (below the half-line) values of λ and σ exist for which the successor function is tangent to the bisectrix, and on the phase plane a corresponding double cycle appears. Such points form a bifurcation curve which emerges from the point λ^+, σ^+ on the half-line $L_1 = 0$.

Tangential contact is impossible when $s_0 < s_0^*$, since not more than one cycle can exist in the combined regions I and II. Therefore, when the parameters are varied, a double cycle emerges at $s_0 = s_0^*$ from the boundary of the region filled with closed trajectories.

3.4. Generation of the limit cycles from the ends of the stationary segment. Let the straight line $L = 0$ and the falling part of the characteristic coincide ($\lambda=\alpha_2$). The falling part of the characteristic will represent the unstable stationary segment and the regions I and II will be filled, by virtue of the condition $(\alpha_1 - 1)^2 < 4\alpha_2$ (see Sect. 2), with the trajectories of the stable foci. We can easily obtain the following explicit expression for the identity transformation of s_0

$$\bar{s}_0 = s_0 \exp(-2h_1\pi / \omega_1) + \delta(\alpha_2 - 1)[1 + \exp(-h_1\pi / \omega_1)]$$

where δ is the width of the region II. The transformation has a single stable stationary point.

Let us now turn the line $L = 0$ around an arbitrary point on the falling segment, in the anticlockwise direction. The stationary segment collapses, a saddle appears in the region II and stable foci in the regions I and III. Let $\lambda = \alpha_2 - \varepsilon$, where $\varepsilon > 0$ and is small. Restricting ourselves to the powers of ε not greater than the first, we obtain the angular coordinates of the separatrices: $[-1 + \varepsilon / (\alpha_2 - 1)]$ for the α-separatrices and $[-\alpha_2 - \varepsilon / (\alpha_2 - 1)]$ for the ω-separatrices. When $\lambda = \alpha_2$, the trajectories emerging from the point at which a saddle appears when $\varepsilon \neq 0$, wind onto the limit cycle. The α-separatrices of the saddle in the region II lie, when $\varepsilon > 0$ and small, in a near neighborhood of the trajectories emerging from the same point at $\varepsilon = 0$, consequently the α-separatrices also wind onto the stable limit cycle which embraces all states of equilibrium. Therefore the ω-separatrices can only be twisted from the unstable cycles lying in the regions I - II and II - III and embracing the stable foci, appearing in the regions I and III, respectively, during the rotation of the straight line. Thus, when the line $L = 0$ rotates, stable foci emerge from the end of the stationary segment embraced by the unstable cycles (the foci and the cycles appear simultaneously). Near each focus we find a unique limit cycle. This follows from the fact that the derivative of the successor function constructed with the help of the saddle-type trajectory in the region II can also be given by the expression (3.3) with a single difference, namely that as s_0 increases, $\theta \to \infty$.

4. Bifurcation of the separatrices. 4.1. Location of the bifurcation curve for the separatrix loop. Let the line $L = 0$ pass, at $\sigma = \sigma_0$ and fixed $\lambda = \lambda^*$, through the upper corner point of the characteristic. We shall vary σ by the amount $\varkappa$ ($\varkappa = \sigma_0 - \sigma$) and show that the separatrix loop cannot arise as the result of the change in the value of σ. Let s_0' and s_1' be the segments intercepted by the α- and ω-separtrices of the linear saddle in the region II on the boundary separating the regions I from II. Let also s_0 and s_1 be the coordinates as defined by the transformation (3.2), on the same boundary. From (3.2) follows

$$s_1 = \delta_0 \chi [\zeta^{-1}(s_0 / \delta_0)] \qquad (4.1)$$

where ζ^{-1} is the inverse of ζ. The quantities h_1 and ω_1 and consequently the functions χ and ζ, are independent of σ.

Since the characteristic is a piecewise linear function, therefore when σ is varied, the quantities s_0', s_1' and δ_0 are proportional to $\varkappa$

$$s_0' = \gamma_0\varkappa, \quad \delta_0 = \gamma_1\varkappa \tag{4.2}$$

$$s_1' = \gamma_2\varkappa \tag{4.3}$$

Matching the trajectories at the boundary between the regions I and II (assuming that $s_0' = s_0$) we obtain from (4.1) and (4.2)

$$s_1 = \gamma_1\varkappa\chi\,[\zeta^{-1}(\gamma_0 / \gamma_1)] \equiv \gamma_3\varkappa \tag{4.4}$$

while from (4.3) and (4.4) we have

$$s_1 / s_1' = \gamma_3 / \gamma_2 = \text{const}$$

Thus for a fixed λ the quantities s_1 and s_1' are in constant ratio and the loop of the separatrix $s_1 = s_1'$ cannot occur when σ varies.

If the line $L = 0$ passes through the middle of the falling segment and $\lambda = \lambda_1$ is such that a separatrix loop exists in the upper part, then by the symmetry of the phase space a separatrix loop must also exist in the lower part. Moreover the condition $\gamma_3/\gamma_2 = 1$ holds. Since γ_3 and γ_2 do not depend on σ, the latter condition holds and both loops are preserved at $\lambda = \lambda_1$ for all values of σ inside the discriminant curve.

4.2. Stability of the separatrix loops. We determine the stability of the separatrix loops by the sign of the saddle parameter when the saddle is situated within or at the boundary of the region II (Theorem 44 of [10] is transposed to the case when the matched loop contains an analytic saddle). In the present case $\alpha_2 > 1$, the saddle parameter is positive $(P_x' + Q_y' = \alpha_2 - 1)$ and the separatrix loops both inside and outside are unstable. On varying the parameters a unique unstable limit cycle either contracts towards the loop or expands away from it (Theorem 47 of [10] is applied to the case when the matched loop contains an analytic saddle).

5. Qualitative structures of the partitioned phase space. 5.1. The phase diagrams corresponding to such values of the parameters σ', λ and σ'', λ, that the lines $\sigma' - \lambda x - y = 0$ and $\sigma'' - \lambda x - y = 0$ which are disposed symmetrically about the middle of the falling segment of the characteristic, are themselves symmetrical with respect to that characteristic. For this reason we can study the partitioning of the phase space by considering only the part of the $\lambda\sigma$ -space that lies either above or below the axis of symmetry $\sigma - \lambda x_0 - y_0 = 0$, where x_0 and y_0 are the coordinates of the middle point of the falling segment.

5.2. Let us investigate the structure of the partition of the phase space and the sequence of bifurcations transforming one structure into another, for the values of the parameters along the line of bifurcation $\sigma - \lambda x_1 - y_1 = 0$ (x_1 and y_1 are the coordinates of the upper corner point of the characteristic).

Let $\lambda > \lambda^+$ (see Fig. 2a). The state of equilibrium is a stable focus on the matching line and all trajectories advance towards it. When $\lambda = \lambda^+$ (Fig. 2b), a region appears, filled with closed trajectories. All trajectories matched over the regions I – III wind onto the boundary of this region. When $\alpha_2 < \lambda < \lambda^+$ (Fig. 2c), the focus of the matching line is unstable and, when the value of λ decreases from $\lambda = \lambda^+$, a stable limit cycle grows from the boundary of the region filled with the closed trajectories. When $\lambda = \alpha_2$ (Fig. 2d) (the edge of the discriminant curve), the falling segment of the characteristic

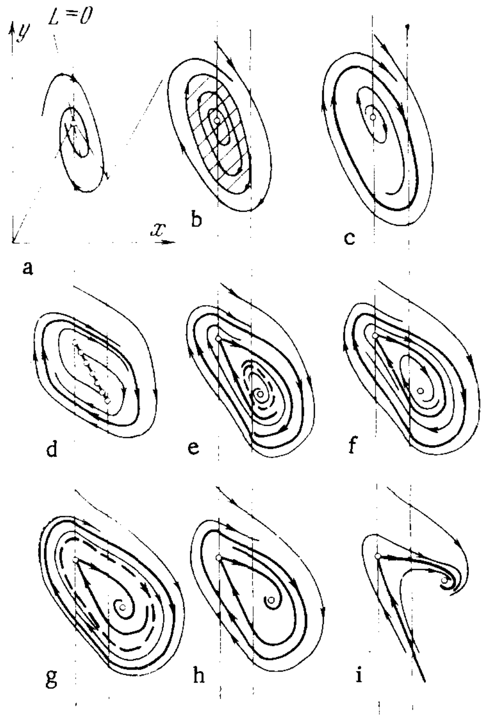

Fig. 2

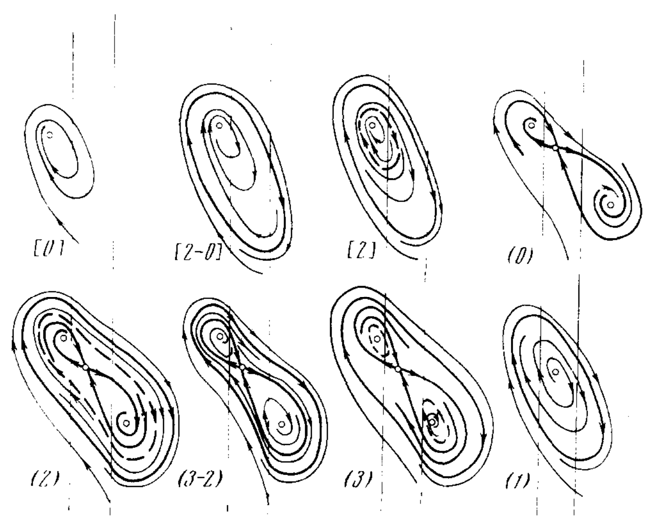

Fig. 3

coincides with the line $L = 0$ and an unstable stationary segment appears within the stable limit cycle. On further decrease in the value of λ along the discriminant curve, two states of equilibrium appear: a matched saddle-focus, and a stable focus in the region III. A focus and an unstable limit cycle appear together from the end of the stationary segment (the α-separatrix of the saddle-focus moves to the stable cycle embracing all states of equilibrium, and the ω-separatrix twists from the unstable cycle embracing the stable focus). Since at $\lambda = 0$ the α-separatrix (line $y = \sigma$) proceeds into the stable node in the region III, the state of equilibrium in the region III remains stable under the variation of parameters along the discriminant curve and the infinity remains unstable, while the limit cycles can vanish over the interval $0 < \lambda < \alpha_2$ only when a merger of the limit cycles takes place followed by supression of the double cycle. This can be realized only due to an intermediate bifurcation, i.e. the appearance at λ $\lambda_1 < \alpha_2$ (Fig. 2e) of a separatrix loop formed from the α- and ω-separatrices of the matched saddle-focus. The separatrix loop is unstable from within and from the outside, and can be regarded as a singular limit cycle with a state of equilibrium on it. It separates the structures with an unstable limit cycle embracing the state of equilibrium in the region III from the structures with an unstable cycle embracing all states of equilibrium.

When the value of λ decreases to $\lambda = \lambda_1$ (Fig. 2f), an unstable limit cycle appears from within the loop and on further decrease in λ and collapse of the loop, an unstable limit cycle grows from the loop (Fig. 2g) and embraces all states of equilibrium (the α-separatrix passes into the stable focus in the region III, the ω-separatrix twists from the unstable limit cycle embracing both states of equilibrium and no states of equilibrium exist between the cycles). At certain $\lambda = \lambda_2 < \lambda_1$ (Fig. 2h) a semistable double limit cycle necessarily appears and vanishes when λ decreases. On further decrease in the value of λ the foci become nodes, and the structure which arises (Fig. 2i) is quantitatively equivalent to the structure at $\lambda = 0$.

5.3. Let us consider the structures within the discriminant curve at $\lambda_1 < \lambda < \alpha_2$. For the values of the parameters belonging to the discriminant curve itself and for the segments intercepted by the α- and ω-separatrices on the matching line, the condition $(s_3)_\alpha > (s_3)_\omega$ holds (the focus is surrounded by an unstable limit cycle) and the inequality cannot be violated at $\lambda = \lambda_0 = \text{const}$ by changing σ. It is preserved, in particular, for the structure at the point of intersection of $\lambda = \lambda_0$ with the axis of symmetry $\sigma - \lambda x_0 - y_0 = 0$ (where x_0 and y_0 are the coordinates of the middle point of the falling segment of the characteristic). At this point the phase diagram is symmetrical with respect to the point x_0, y_0, consequently we also have an unstable limit cycle surrounding the stable focus in the region I. This qualitative pattern inside of the region bounded by the discriminant curve cannot be affected by the change in σ. This implies that, when one moves away from the discriminant curve in the inward direction and the matched state of equilibrium of the saddle-focus type is disrupted, then a saddle appears in the region II and a stable focus in the region I, accompanied by an unstable limit cycle.

The structure of the partitioning of the phase space at $\lambda_1 < \lambda < \alpha_2$ will contain three limit cycles; α-separatrices of the saddle move towards a stable cycle embracing all three states of equilibrium, the ω-separatrices twist from the unstable cycles embracing the foci in the regions I and III. When $\lambda = \lambda_1$, the upper and lower separatrix loops are formed simultaneously and unstable limit cycles appear in them. When λ decreases from $\lambda = \lambda_1$, an unstable limit cycle appears from the double loop and embraces all states of equilibrium.

Since when $\lambda < {}^1/_4\,(\alpha_1 - 1)^2$ the system has no limit cycles (regions I and III are "crossed out" by the integral straight lines of the linear nodes), the stable and unstable cycles must vanish on the interval ${}^1/_4\,(\alpha_1 - 1)^2 < \lambda < \lambda_1$ (see Fig. 2i). A bifurcation curve of the double cycles must exist inside the discriminant curve to the left of the segment of the separatrix loops. By virtue of the symmetry mentioned previously, the curve of double cycles intersects the sides of the discriminant curve at one and the same value of $\lambda = \lambda_2$.

5.4. The bifurcation curve corresponding to the merger of the stable and unstable cycles (double cycle curve) begins at the points at which $\lambda = \lambda^+$ on the lines $L_1 \equiv \sigma - \lambda x_1 - y_1 = 0$ and $L_2 \equiv \sigma - \lambda x_2 - y_2 = 0$ and is situated in the case of a single state of equilibrium, respectively, below and above the two straight lines $L_1 = 0$ and $L_2 = 0$ (in the region $L_1 L_2 > 0$). When $\lambda = \lambda_2$, both branches of the double cycle curve intersect (below and above) the discriminant curve and pass into each other within the discriminant curve.

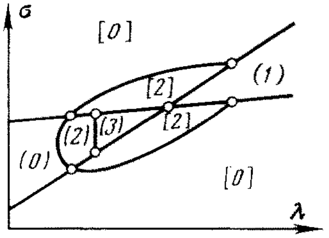

Fig. 4

The double cycle curve separates, on the interval $\lambda_2 < \lambda < \lambda^+$ in the region $L_1 L_2 > 0$, a region near $L_1 = 0$ and $L_2 = 0$ and for the points belonging to this region we have one stable state of equilibrium and two limit cycles in the phase space.

5.5. If the focus-type state of equilibrium lies on the falling segment of the characteristic (region $L_1 L_2 < 0$, $\lambda > \alpha_2$ on the parameter plane), then a unique stable limit cycle exists. The existence of at least one cycle is obvious (the state of equilibrium and the infinity are unstable). The uniqueness follows from the fact that the exponential index in (3.3) and (3.4) varies monotonously, and this in turn is caused by the fact that the focus lies on the falling segment of the characteristic (when s_0 increases, the parameters τ_1 and τ_3 also increase, while τ_2, τ_4 and θ decrease) and, that the successor function is differentiable at the matching point.

5.6. Figure 4 shows how the $\lambda\sigma$-parameter space is divided into regions with different qualitative structure of the phase space. The numbers in the round and square brackets (the figures indicate the number of limit cycles) denote the regions. The sequence of the qualitative structures along the bifurcation line $L_1 = 0$ is shown in Fig. 2. Figure 3 uses the same numbering system as Fig. 4 to designate the coarse qualitative structures corresponding to the various regions of the phase space. The fine structures in Fig. 3 denoted by the double numbers correspond to the bifurcation curves in Fig. 4, separating the relevant regions.

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TWO-FREQUENCY RESONANT OSCILLATIONS OF CONSERVATIVE SYSTEMS

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Two-frequency oscillations of a conservative system with n degrees of freedom are studied. The problem is reduced to investigate canonical systems describing resonance phenomena. Single-frequency and multi-frequency oscillations were studied earlier in [1 – 3].

1. Let us consider a conservative system with n degrees of freedom which has a stable state of equilibrium and executes relatively small motions in the neighborhood of this state. The differential equations of motion of the system have the form

$$\sum_{k=1}^{n} (a_{ik}q_k^{\cdot\cdot} + c_{ik}q_k) = - \sum_{k,j=1}^{n} [a_{ik}^{(j)}(q_k q_j^{\cdot\cdot} + \frac{1}{2} q_k^{\cdot} q_j^{\cdot}) + \frac{1}{2} c_{ik}^{(j)} q_k q_j] - \tag{1.1}$$
$$\sum_{k,j,s=1}^{n} \left[\frac{1}{2} a_{ik}^{(js)} (q_k q_j q_s^{\cdot\cdot} + q_k q_j^{\cdot} q_s^{\cdot}) + \frac{1}{6} c_{ik}^{(js)} q_k q_j q_s\right] - \ldots \quad (i = 1, 2, \ldots n)$$

Let the system undergo two-frequency oscillations of frequencies ω_1 and $\omega_2 (\omega_2 \geqslant \omega_1)$. We shall consider the two-frequency resonant solutions of the system (1.1). We define the degree of the resonance terms in the right-hand sides of (1.1), as the resonance rank. The ratio ω_2 / ω_1 for these terms is essential.